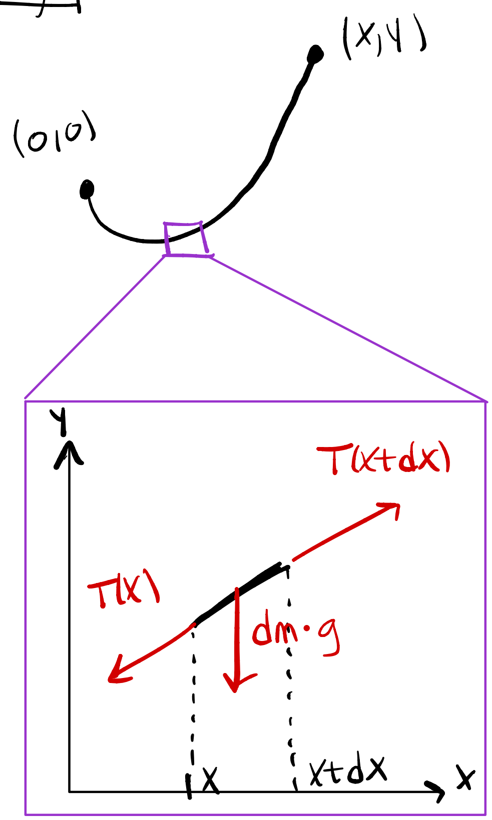
Catenary | 2022/10/02

What is the shape of a rope of mass m and length ℓ?



The rope is stationary:

$$\begin{cases} T_x(x+dx) - T_x(x) = 0 \\ T_y(x+dx) - T_y(x) - dm\cdot g = 0 \end{cases}$$

$$\frac{dT}{dx} = \frac{T_x(x+dx) - T_x(x)}{dx} = 0$$

$$\longrightarrow T_x(x) = \text{const.} \equiv c$$

what is dm? We know that

$$ds = \sqrt{dx^2+dy^2} = dx\sqrt{1+y'(x)^2}$$

$$dm = \left(\frac{m}{\ell}\right) ds = \frac{m}{\ell} dx\sqrt{1+y'(x)^2}$$

$$\downarrow$$

$$T_y(x+dx) - T_y(x) - \frac{mg}{\ell} dx\sqrt{1+y'(x)^2} = 0$$

we also know that $\vec{T}$ is tangent to $y(x)$. Therefore

$$y'(x) = \frac{T_y(x)}{T_x(x)} \longrightarrow T_y(x) = T_x(x)\, y'(x)$$

$$\longrightarrow T_y(x+dx) - T_y(x) - \frac{mg}{\ell} dx\sqrt{1+y'(x)^2} = 0$$

$$c\left[y'(x+dx) - y'(x)\right] - \frac{mg}{\ell} dx\sqrt{1+y'(x)^2} = 0$$

$$c\,y''(x) - \frac{mg}{\ell}\sqrt{1+y'(x)^2} = 0$$

$$y''(x) = K\sqrt{1+y'(x)^2} \; ; \; K = \frac{mg}{c\ell}$$

we solve the equation above for $y(x)$

$$y''(x) = K\sqrt{1+y'(x)^2} \xrightarrow[\text{let } u = y'(x)]{} \frac{du}{dx} = K\sqrt{1+u^2}$$

$$\underbrace{\int \frac{du}{\sqrt{1+u^2}}}_{} = \int K\,dx$$

let $v = iu$

$du = -i\,dv$

$$\int \frac{-i\,dv}{\sqrt{1-v^2}} = -i\sin^{-1}(v) = -i\sin^{-1}(iu)$$

$$\longrightarrow -i\sin^{-1}(iu) = Kx + A$$

$$iu = \sin\left(\frac{Kx+A}{-i}\right) = \frac{e^{-(Kx+A)} - e^{+(Kx+A)}}{2i}$$

$$u = \frac{e^{(Kx+A)} - e^{-(Kx+A)}}{2} = \sinh(Kx+A)$$

$$\longrightarrow y'(x) = \sinh(Kx+A)$$

we integrate to find $y(x)$

$$\int \frac{dy}{dx} = \int \frac{1}{2}\left[e^{(Kx+A)} - e^{-(Kx+A)}\right]$$

$$y(x) = \frac{1}{2}\left(\frac{1}{K}\right)\left[e^{(Kx+A)} + e^{-(Kx+A)}\right] + B$$

$$\boxed{y(x) = \frac{\cosh(Kx+A)}{K} + B \;;\; K = \frac{mg}{\ell}C}$$

we apply boundary conditions to find A, B, C. (we assume the endpoints of the rope are (x_1, y_1) and (x_2, y_2))

$$\begin{cases} y(x_1) = \frac{1}{K}\cosh(Kx_1 + A) + B = y_1 \\ y(x_2) = \frac{1}{K}\cosh(Kx_2 + A) + B = y_2 \\ \int ds = \int_{x_1}^{x_2} dx \underbrace{\sqrt{1+y'(x)}}_{y''(x)/K} = \frac{1}{K}y'(x)\Big|_{x_1}^{x_2} = \frac{1}{K}\left[\sinh(Kx_2+A) - \sinh(Kx_1+A)\right] = \ell \end{cases}$$

These equations are generally solved numerically.